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CONVECTION IN THE FREE ATMOSPHERE AND OVER A HEATED SURFACE

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In a previous contribution to this journal (*1*) the author discussed a set of differential equations which, under certain assumptions, enables us to compute the distribution of atmospheric eddy energy at any time, this distribution being given at the time $t=0$. A few integrals were derived in order to illustrate the application of the theory under different specified conditions. One of the examples referred to the overturning at the horizontal boundary between two air layers in convective equilibrium, the upper having a slightly lower potential temperature than the other. In view of the important rôle this kind of convection plays over the North American Continent (*2*) it seemed justified to devote more attention to its theoretical side. As a result, an integral was obtained, which determines mathematically the overturning also for the case, where each of the two superposed air layers, taken separately, is in stable equilibrium.

The method by which this solution was obtained was then applied to the problem of local convection and gave an integral which in a general way describes the rise of convection currents from a heated surface.

Finally, the differential equations of convection were generalized in order to account for temperature changes produced by radiation.

Numerous references will be made to the formulæ in the original paper (*1*), these references always being in the form (*1*, II, 13) where *1* signifies the paper, II the section, and 13 the number of the formula. The mathematical symbols used below have the same meaning as in (*1*).

CONVECTION IN THE FREE AIR

Suppose that, within a body of air (for instance 6 kilometers high and 100 kilometers square) the potential temperature (θ) is distributed according to the law

$$\theta=\theta_{\mathrm{m}}+\beta h \tag{1}$$

where β, the rate at which θ increases upward, is a positive constant. The horizontal plane of reference $h=0$ may be assumed to be situated somewhere in the free atmosphere, say 3 kilometers above sea level.

Now assume that increasing southerly winds in the lower layers bring warmer air masses in below the plane $h=0$. At the same time, northerly winds bring in colder air above. If the new air masses have the same degree of stability as the original, then, in the absence of overturning, we should have, for positive h-values,

$$\theta=\theta_{\mathrm{m}}-\Delta+\beta h, \tag{2}$$

and for negative values

$$\theta=\theta_{\mathrm{m}}+\Delta+\beta h \tag{3}$$

Thus the level $h=0$ is characterized by a sudden temperature drop of 2Δ. Now it is obvious that such a sudden drop can never persist for any length of time. Immediately the colder air mass is brought in above the plane $h=0$, turbulence and overturning will set in and spread the temperature drop 2Δ over a small vertical distance. However, in order to facilitate the theoretical treatment, we shall here assume that at the time $t=0$ the potential temperature is distributed according to (2) and (3) and that at this moment the overturning begins. Let the thickness of the intermediate turbulent layer at any time be $2H$; H therefore is a function of t to be determined later. The turbulence within the layer $2H$ is accompanied by heat convection. The simultaneous changes in the distribution of kinetic eddy energy per unit mass (E) and potential temperature (θ) are interrelated through the two equations (*1*, II, 53). As pointed out in (*1*), solutions of these equations are not easily obtained. For the special case $\beta=0$ (convective equilibrium in the undisturbed airmasses) a simple integral was, however, derived. In the general case $\beta>0$ this solution is not valid; θ will no longer, as in (*1*, II, 66), follow a simple linear law within the turbulent layer but has a more complicated analytical form. If, however, we assume that the turbulent mixing is strong enough to produce an approximate linear distribution of θ, the second equation (*1*, II, 53) can be disregarded. The distribution of E between $h=-H$ and $h=+H$ is then determined solely by

$$\rho\frac{\delta E}{\delta t}=a\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right]-cg\frac{E}{\theta}\frac{\delta\theta}{\delta h} \tag{4}$$

Within the same layer θ, according to our assumption, is given by

$$\theta=\theta_{\mathrm{m}}-\alpha h \tag{5}$$

Since θ must be continuous for $h=+H$, we have

$$\theta_{\mathrm{m}}-\Delta+\beta H=\theta_{\mathrm{m}}-\alpha H \tag{6}$$

or

$$\alpha=-\frac{\delta\theta}{\delta h}=\frac{\Delta-\beta H}{H} \tag{7}$$

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If the variations of θ in the last number of (4) are neglected and the constant

$$q=\frac{c \cdot g}{a \cdot \theta_m} \tag{8}$$

is introduced, then (4) reduces to

$$\frac{\delta E}{\delta \tau}=\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right]+q\alpha E \qquad \left(\tau=\frac{at}{\rho}\right) \tag{9}$$

Now assume for E a solution of a type already discussed in (*1*), namely

$$E=k(H^2-h^2) \tag{10}$$

If we denote differentiation with respect to τ by a prime ($'$), then we must have

$$(kH^2)'-k'h^2=-2k^2H^2+6k^2h^2+q\alpha k(H^2-h^2) \tag{11}$$

Since this equation must hold for any $h-$values,

$$(kH^2)'=-2k^2H^2+q\alpha kH^2 \tag{12}$$

and

$$-k'=6k^2-q\alpha k \tag{13}$$

Multiplying the second equation by H^2 and adding it to the first, we obtain

$$(H^2)'=4kH^2 \tag{14}$$

or

$$k=\frac{1}{2H}\cdot H' \tag{15}$$

Introducing the expressions for k and α in (13), we obtain

$$-\frac{1}{2}\left(\frac{H'}{H}\right)'=\frac{3H'^2}{2H^2}-q\frac{\Delta-\beta H}{H}\cdot\frac{1}{2H}\cdot H' \tag{16}$$

This equation is simplified to

$$H'^2-H\cdot H''=3H'^2-q(\Delta-\beta H)\cdot H' \tag{17}$$

Writing

$$H''=H'\cdot\frac{dH'}{dH} \tag{18}$$

we transform (17) to

$$\frac{dH'}{dH}+\frac{2}{H}\cdot H'=q\frac{(\Delta-\beta H)}{H} \tag{19}$$

The integral of this equation is

$$H'=\frac{A}{H^2}+\frac{q\Delta}{2}-q\beta\cdot\frac{H}{3} \qquad (A=\text{const.}) \tag{20}$$

If we assume that the velocity of the overturning is finite at $t=0$, then A must vanish. Thus

$$H'=\frac{q\Delta}{2}-\frac{q\beta}{3}H \tag{21}$$

and

$$H=\frac{3\Delta}{2\beta}\left(1-e^{-\frac{q\beta\tau}{3}}\right). \tag{22}$$

The turbulent layer will therefore never exceed the limits $\pm H_\infty=\pm\frac{3\beta}{2\Delta}$. The more stable the original stratification, the less the turbulent layer will spread.

The maximum value of E (for $h=0$) is at any time given by

$$E_0=kH^2=\frac{3q\Delta^2}{8\beta}\,e^{-\frac{q\beta\tau}{3}}\left(1-e^{-\frac{q\beta\tau}{3}}\right) \tag{23}$$

This function is at a maximum when

$$e^{-\frac{q\beta\tau}{3}}=\frac{1}{2} \tag{24}$$

Then

$$H=\frac{3\Delta}{4\beta} \text{ and } \alpha=\frac{\beta}{3} \tag{25}$$

Integrating (10) between the limits $\pm H$ we obtain an expression for the total eddy energy,

$$E_{\text{Total}}=\frac{4\rho kH^3}{3}$$

By means of (15) and (21) E_{Total} can be expressed as a function of H (and thus of τ). We find, as would be anticipated, that the total eddy energy will increase, until the temperature lapse rate within the turbulent layer has reached the adiabatic. At that moment all the available potential energy of the stratification has become converted into kinetic energy. From then on, the eddies, continuing to diffuse upward and downward while the lapse rate decreases, work against gravity, transforming kinetic energy into heat and potential energy. After a theoretically infinitely long time, when the convective layer has reached the limits $\pm H_\infty$, this latter energy conversion is completed and the atmosphere again at rest.

CONVECTION OVER A HEATED SURFACE

Now let us try to obtain a solution of the following problem: An air-mass is originally at rest and in stable equilibrium. The potential temperature is distributed according to the formula

$$\theta=\theta_0+\beta h, \tag{26}$$

where β is a positive constant and h the height above ground. Suddenly (for $t=0$) the surface is heated to $\theta_0+\Delta$ and then kept at a constant temperature. The air layer close to the surface will immediately assume the temperature $\theta_0+\Delta$. Thus the lowest layers will be characterized by a superadiabatic lapse rate, which will give rise to turbulent overturning and convection currents. Within the turbulent layer the lapse of potential temperature will decrease gradually from a strong positive value close to the ground. To simplify the problem we shall, however, in this case also, assume that θ within the convective layer is distributed according to a linear law,

$$\theta=\theta_0+\Delta-\alpha h.\text{[1]} \tag{27}$$

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[1] It is easily proved that this assumption can only be approximately true. The convection current of heat through any level is determined by

$$-c_p.c.E\cdot\frac{\delta\theta}{\delta h},$$

where c_p signifies the specific heat at constant pressure. In case of surface convection there is a steady flow of heat from the earth to the atmosphere. The above expression does therefore in general not vanish for $h=0$. Thus we must have, in the vicinity of ground,

$$E\frac{\delta\theta}{\delta h}=\mu+\nu h+\ldots \qquad (\mu\neq 0)$$

Now we have proved (*1*, III, 3), that E close to the surface can be developed in the form

$$E=\kappa\sqrt{h}\,(1+\text{terms of higher order})$$

Thus, for small $h-$values,

$$\frac{\delta\theta}{\delta h}=\frac{\mu}{\kappa\sqrt{h}}+\frac{\nu}{\kappa}\sqrt{h}+\ldots$$

or

$$\theta=\text{Constant}+\frac{2\mu}{\kappa}\sqrt{h}+\ldots.$$

θ in the surface layer therefore varies proportional to $\sqrt{h}$.

To compute α we have, since θ must be continuous,

(28) $$\theta_0+\Delta-\alpha H=\theta_0+\beta H$$

or

(29) $$\alpha=\frac{\Delta-\beta H}{H}$$

H is the height of the turbulent layer.

In (*1*, III) we have proved that E close to ground must be proportional to $\sqrt{h}$. Thus it is most natural to introduce a function $P(x)$ by

(30) $$E=x.\ P(x),$$

where

(31) $$x=\sqrt{h}$$

The equation (9) then changes into

(32) $$x\frac{\delta P}{\delta\tau}=\frac{1}{4x}\frac{\delta}{\delta x}\left[P\frac{\delta(xP)}{\delta x}\right]+\alpha q\cdot x\cdot P$$

or

(33) $$4x^2\frac{\delta P}{\delta\tau}-4\alpha qx^2P=\frac{\delta}{\delta x}\left[P\frac{\delta(xP)}{\delta x}\right]$$

Assume for P a solution of the form

(34) $$P=a-bx^3$$

where a and b are functions of τ. If this expression is inserted in (33) we obtain

(35) $$(4a'-4\alpha qa)x^2-(4b'-4\alpha qb)x^5=-15abx^2+24b^2x^5$$

a and b must therefore fulfill the equations

(36) $$a'-\alpha qa=-\frac{15}{4}\,ab$$

(37) $$b'-\alpha qb=-6b^2$$

Multiplying the first of these equations by b, the second by a and then subtracting, we obtain

(38) $$a'b-ab'=-\frac{15ab^2}{4}+6ab^2$$

or

(39) $$\left(\frac{a}{b}\right)'=\frac{9a}{4}$$

Since E, and hence also P, must vanish for $h=H$ or $x=\sqrt{H}$, we have

(40) $$\frac{a}{b}=H^{\frac{3}{2}}$$

Thus, from (39) it follows that

(41) $$a=\frac{2}{3}\sqrt{H}\cdot H'$$

(42) $$b=\frac{2}{3}\ \frac{1}{H}\cdot H'$$

Inserting the expression for b in (37) we obtain, after certain reductions,

(43) $$HH''-q(\Delta-\beta H)\ H'+3\ H'^2=0$$

Changing the independent variable we may write

(44) $$H''=H'.\ \frac{\delta H'}{\delta H}$$

The equation (43) then takes the form

(45) $$H\frac{\delta H'}{\delta H}+3H'=q\,(\Delta-\beta H),$$

which can be integrated and gives

(46) $$H'=\frac{A}{H^3}+\frac{q\Delta}{3}-\frac{q\beta H}{4}\qquad\left(A=\text{const.}\right)$$

Since H' is finite for $H=0$, we must have $A=0$. Integrating once more we obtain

(47) $$H=\frac{4}{3}\ \frac{\Delta}{\beta}\left(1-e^{\frac{-q\beta\tau}{4}}\right)$$

Using (41), (42), and (34), we write tho solution for E in the form

(48) $$E=\frac{2}{3}\ \frac{H'}{H}\ \sqrt{h}\ (H^{\frac{3}{2}}-h^{\frac{3}{2}})$$

The equations (47) and (48) constitute the solution of our problem.

The maximum value of E is found at a given time the height

(49) $$\frac{H}{2\cdot\sqrt[3]{2}}=0.39H$$

After a few simplifications we find that $E_{\max}$ has the form

(50) $$E_{\max}=\frac{H\cdot H'}{2\cdot 2^{\frac{2}{3}}}$$

This function of τ has its maximum value for

(51) $$e^{\frac{-q\beta\tau}{4}}=\frac{1}{2}$$

Then

(52) $$H=\frac{2}{3}\ \frac{\Delta}{\beta}$$

and

(53) $$\alpha=\frac{\beta}{2}$$

Thus $E_{\max}$ reaches its greatest value already before the lapse rate has become adiabatic. The reason for this is obviously to be found in the increasing dissipation of eddy energy at the surface. This dissipation, when H passes a certain value, will exactly balance the production of eddies, which decreases with the decreasing instability of the convective layer.

An example of the solution (48) has been computed and plotted in the figure. The following numerical constants have been used:

$$\frac{c}{a}=1,\ q=\frac{g}{\theta_0}=3.3,\ \beta=2\cdot 10^{-5},\ \Delta=6\ \text{centigrades.}$$

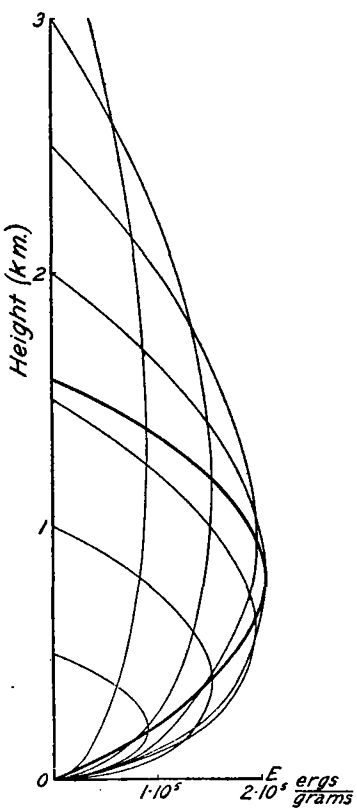
Under these assumptions it is found that the maximum height reached by convection is the 4 kilometers level. In the figure the thin lines give the distribution of E from time to time, while the thick line shows the traveling of $E_{\max}$. It might be worth emphasizing that the solution (48) is independent of the numerical value of the constant a and contains only the ratio $\frac{c}{a}$, to which it is directly proportional. The constant a occurs only as multiplied by t in (47). A good determination of a is therefore necessary if we wish to determine the velocity with which convection rises.

Now, assume that the atmosphere was in indifferent equilibrium ($\beta=0$) until the time $t=0$, at which moment the surface was heated Δ degrees. Convection will then start and it may be assumed that E approximately follows the law just derived,

$$E=\frac{2}{3}\sqrt{h}\,\frac{H'}{H}\left(H^{\frac{3}{2}}-h^{\frac{3}{2}}\right) \tag{54}$$

In this special case H has the value

$$H=\frac{q\Delta}{3}\tau, \quad H'=\frac{q\Delta}{3} \tag{55}$$



It is now possible to compute, under the assumption just made, the temperature distribution within the convective layer. According to (1, II, 53b)

$$\frac{\delta\theta}{\delta\tau}=\frac{\delta}{\delta h}\left[E\frac{\delta\theta}{\delta h}\right] \tag{56}$$

In this equation we have, for sake of simplicity, assumed $c=a$. One finds as a solution of this equation

$$\theta=\theta_0+\Delta-\sqrt{\frac{h}{H}}\left(\Delta+\frac{\sigma}{H^2}\right)+\sigma\cdot\frac{h^2}{H^4} \tag{57}$$

Here we have denoted the constant of integration by σ.

As earlier pointed out, the transport S of heat from the ground to the air is at any moment given by

$$S=-c_p\left(cE\frac{\delta\theta}{\delta h}\right)_{h=0}, \tag{58}$$

or, from (54), (55), and (57),

$$S=c_p\cdot c\cdot\frac{q\Delta}{9}\left(\Delta+\frac{\sigma}{H^2}\right) \tag{59}$$

Since this transport can not be infinite at the time $t=0$, the constant σ must obviously vanish. Thus we obtain

$$\theta=\theta_0+\Delta-\sqrt{\frac{h}{H}}\cdot\Delta \tag{60}$$

and

$$S=c_p\cdot c\cdot\frac{q\Delta^2}{9} \tag{61}$$

From (55) and (61) it is theoretically possible to compute the flow of heat that produces a convection current of given intensity. However, the following must be kept in mind. In deriving the temperature distribution (60) we made use of the equation (56), according to which all temperature changes are due to convective transport of heat. Differences between absorbed and emitted radiation are thus not taken into account. As a matter of fact, part of the convection-producing flow of heat is conveyed to the atmosphere not through surface conduction, but through excess absorption, in the lowest air layers, of the long wave radiation from the surface. The more evenly this absorption is distributed throughout a thick layer, the less effective it will be in creating convection currents. The solution here presented therefore rather gives an upper limit for the convection produced by a given flow of heat. Writing (55) in the form

$$\frac{dH}{dt}=\frac{q\Delta}{3}\cdot\frac{a}{\rho} \tag{62}$$

we can eliminate Δ between this formula and (61). The result is

$$\frac{dH}{dt}=\frac{1}{\rho}\sqrt{\frac{aq}{c_p}}\cdot\sqrt{S} \tag{63}$$

Thus the rate at which the convection current rises is proportional to the square root of the flow of heat conveyed to the lowest air layer.

It was shown in (1), that the constant a (or c) generally is of the order of magnitude 10^{-3} to $^{-2}$, and in a numerical calculation then made a was assumed to be equal to 10^{-3}. This later value seems, however, somewhat low. To obtain an idea of the magnitude of the quantities discussed let us give a numerical example illustrating (61) and (62). We assume

$$q=3.3,$$
$$\Delta=4 \text{ centigrades}$$
$$\rho=10^{-3}$$
$$c_p=0.24$$

Then

$$\frac{dH}{dt}=158.4\cdot10^3\,a\,\frac{\text{meters}}{\text{hour}},\quad S=84.48\,c\,\frac{\text{gm. cal.}}{\text{cm.}^2/\text{min.}}$$

Assuming in these expressions a value for a (or c) about half-way between the limits given above, $a=c=4.10^{-3}$, we find for $\frac{dH}{dt}$ and S the numerical values

$$\frac{dH}{dt}=634\,\frac{\text{meters}}{\text{hour}},\quad S=0.34\,\frac{\text{gm. cal.}}{\text{cm.}^2/\text{min.}}$$

If the surface of the earth radiates as a black body then it would at a temperature of 27° C or 300° absolute send out 0.67 $\frac{\text{gm. cal.}}{\text{cm.}^2/\text{min.}}$ Thus, according to the above example, the heat carried off by convection is about 50 per cent of the amount sent out through radiation during the same time. The rate at which the computed convection current rises seems to agree well with what we know about the growth of cumulus.

In the general case, where the eddy-producing flow of heat is transferred to the atmosphere partly through radiation, the equation (*1*, II, 53b) has to be suitably modified. The relation between absolute temperature (T) and potential temperature (θ) is, as well known,

$$T=\theta\cdot\left(\frac{p}{P_0}\right)^{\frac{\kappa-1}{\kappa}} \tag{64}$$

Here p is the actual pressure, P_0 the standard pressure and κ the ratio between the two specific heats. The exponent $\frac{\kappa-1}{\kappa}$ has the numerical value 0.29. Multiplying equation (1, II, 53b) by $\left(\frac{p}{P_0}\right)^{0.29}$ and assuming that the pressure remains constant, we obtain with the aid of (64)

$$\rho\,\frac{\delta T}{\delta t}=c\cdot\left(\frac{p}{P_0}\right)^{0.29}\frac{\delta}{\delta h}\left[E\,\frac{\delta\theta}{\delta h}\right] \tag{65}$$

If this equation is multiplied by c_p, the left member gives the amount of heat per unit volume needed to increase the temperature of the air $\frac{\delta T}{\delta t}$ per second. According to (65) this heat is furnished solely by convection. If, however, per unit time and mass the air absorbs Δ calories more than it emits, then we have to add to the right member the term $\rho\Delta$. Thus

$$c_p\rho\,\frac{\delta T}{\delta t}=c\cdot c_p\cdot\left(\frac{p}{P_0}\right)^{0.29}\frac{\delta}{\delta h}\left[E\,\frac{\delta\theta}{\delta h}\right]+\rho\Delta \tag{66}$$

or

$$\rho\,\frac{\delta\theta}{\delta t}=c\cdot\frac{\delta}{\delta h}\left[E\,\frac{\delta\theta}{\delta h}\right]+\frac{\rho}{c_p}\cdot\left(\frac{P_0}{p}\right)^{0.29}\cdot\Delta \tag{67}$$

The equations determining atmospheric convection now take the more general form

$$\rho\,\frac{\delta E}{\delta t}=a\,\frac{\delta}{\delta h}\left[E\,\frac{\delta E}{\delta h}\right]-c\cdot\frac{g}{\theta}\,E\,\frac{\delta\theta}{\delta h} \tag{68}$$

$$\rho\,\frac{\delta\theta}{\delta t}=c\,\frac{\delta}{\delta h}\left[E\,\frac{\delta\theta}{\delta h}\right]+\frac{\rho}{c_p}\cdot\left(\frac{P_0}{p}\right)^{0.29}\cdot\Delta \tag{69}$$

The quantity Δ, being the difference between emission and absorption, is a function of T and hence also of θ. A complete solution of the convection problem is possible only if to the equations (68) and (69) are added Schwarzschild's differential equations for the radiation currents penetrating the atmosphere. In such a general formulation however, the problem presents insurmountable mathemathical difficulties.

If the last term in (69) is a known function of the height then the equations are easily integrated for the stationary case, when $\frac{\delta E}{\delta t}$ and $\frac{\delta\theta}{\delta t}$ vanish. We leave that to the reader.

Recently L. Keller (*3*) has made an extremely interesting attempt to derive a complete system of characteristics for atmospheric turbulence; in other words, a system of quantities which, with a sufficient degree of accuracy, describe statistically this turbulence, and of such a nature that if given at the time $t=0$ they may be computed for any subsequent time. Keller's attempt is based upon a generalization of Osborne Reynold's method for deriving the additional stresses within a turbulent moving fluid. It is applied to the case of adiabatic movement. The problem can be solved only under certain simplifying assumptions and then leads to a system of 35 functional-differential equations of a rather complicated nature.

What we have done in this and the preceding paper is obviously to reduce the characteristics of atmospheric turbulence for the case of pure convection, with no horizontal mean wind components, to two quantities, E and θ. To be sure, the way in which we have made this reduction may, from a purely theoretical point of view, be seriously criticized. It seems, however, from the integrals already derived that the theory is very fertile and in a general way well describes several phenomena related to atmospheric turbulence. Little attention need be paid to the numerical values arrived at in this as well as in the preceding paper. The quantitative results are dependent upon several rather arbitrary assumptions. Their validity or nonvalidity does not in any way affect the applicability of the theory in its general form.

It is extremely desirable that parallel with this semi-empirical line of work an attack be made on the turbulence problem from the pure theoretical side in the sense of Keller's study. A confrontation of the two theories may lead to the introduction of new and useful conceptions.

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LITERATURE CITED

(*1*) ROSSBY, C.-G.
1926. THE VERTICAL DISTRIBUTION OF ATMOSPHERIC EDDY ENERGY. Monthly Weather Review, August, 1926.

(*2*) ROSSBY, C.-G. and WEIGHTMAN, R. H.
1926. APPLICATION OF THE POLAR FRONT THEORY TO A SERIES OF AMERICAN WEATHER MAPS. Monthly Weather Review, December, 1926.

(*3*) KELLER, L.
1925. ÜBER DIE AUFSTELLUNG EINES SYSTEMS VON CHARAKTERISTIKEN DER ATMOSPHÄRISCHEN TURBULENZ Journal of Geophysics and Meteorology, Vol. II No. 3-4.